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An $SL(2, \mathbb{Z})$ anomaly in IIB supergravity and its F-theory interpretation

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Abstract: The $SL(2, \mathbb{Z})$ duality transformations of type IIB supergravity are shown to be anomalous in generic F-theory backgrounds due to the anomalous transformation of the phase of the chiral fermion determinant. The anomaly is partially cancelled provided the ten-dimensional type IIB theory lagrangian contains a term that is a ten-form made out of the composite $U(1)$ field strength and four powers of the curvature. A residual anomaly remains uncancelled, and this implies a certain topological restriction on consistent backgrounds of the euclidean theory. A similar, but slightly stronger, restriction is also derived from an explicit F-theory compactification on $K_3 \times M^8$ (where M^8 is an eight-manifold with a nowhere vanishing chiral spinor) where the cancellation of tadpoles for Ramond–Ramond fields is only possible if M^8 has an Euler character that is a positive multiple of 24. The interpretation of this restriction in the dual heterotic theory on $T^2 \times M^8$ is also given.

Keywords: Superstrings and Heterotic Strings, F-Theory, D-branes, Anomalies in Field and String Theories.

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Contents

1. Introduction

The unravelling of the discrete local gauge symmetries of string theory may provide significant clues concerning its underlying geometric structure. We will here be concerned with how the $SL(2,\mathbb{Z})$ symmetry of type IIB string theory is consistently realized upon compactification.

Classical type IIB supergravity [1, 2, 3, 4] is invariant under a global $SL(2,\mathbb{R})$ symmetry. In a gauge invariant formulation of the theory there are three scalar fields that parametrise an $SL(2,\mathbb{R})$ matrix while the chiral fermionic fields transform under a local $U(1)$ (or $O(2)$) symmetry but are inert under the global $SL(2,\mathbb{R})$ transformations. Upon fixing the gauge the $U(1)$ is identified with a subgroup of $SL(2,\mathbb{R})$ and one of the scalars is eliminated. The two remaining scalars parametrise the coset space $SL(2,\mathbb{R})/U(1)$, or the upper-half plane. In this case a general $SL(2,\mathbb{R})$ transformation must be accompanied by a compensating $U(1)$ gauge transformation in order to maintain the gauge condition. This induces a nontrivial $SL(2,\mathbb{R})$ transformation on the fermion fields. Of course, the consistency of this procedure requires that the $U(1)$ symmetry is not anomalous. The structure of the IIB theory and the gauge fixing procedure will be reviewed in section 2.

The continuous $SL(2,\mathbb{R})$ symmetry of the classical supergravity does not survive in the quantum theory since it is not preserved in the string extension of type IIB supergravity. Indeed it is well known [5] that the classical superstring is not invariant under the $U(1)$ subgroup of $SL(2,\mathbb{R})$ that rotates the two supercharges into each other. Since the two supercharges move in opposite directions on the world-sheet the theory is only invariant under a discrete $\mathbb{Z}_4$ subgroup of this $U(1)$ which interchanges the two supercharges *and* reverses the direction σ. It is also well-established by now that the $SL(2,\mathbb{R})$ symmetry of classical IIB supergravity is actually replaced by a discrete $SL(2,\mathbb{Z})$ *local* symmetry in string theory [6]. Thus, the moduli space of the scalar fields becomes $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)$ and the $\mathbb{Z}_4$ symmetry that acts on the supercharges is the intersection of $U(1)$ with $SL(2,\mathbb{Z})$.

In this paper we analyse an anomaly in the action of the $SL(2,\mathbb{Z})$ duality symmetry group. This anomaly can be understood to originate from an anomaly in the conservation of the local chiral $U(1)$ current that will be exhibited in section 3.[1] There we will evaluate an anomalous hexagon diagram coupling the divergence of the current to four gravitons and to the current itself. As expected, the anomaly can be cancelled by the addition of a local counterterm, S', to the action but S' is not invariant under $SL(2,\mathbb{Z})$ transformations. The result is that the $SL(2,\mathbb{Z})$ transformations are anomalous for generic backgrounds in which the complex scalar field is not constant – *i.e.*, in F-theory backgrounds. The $SL(2,\mathbb{Z})$ transformation of the counterterm, $\delta S'$, is given by an integral that we will evaluate explicitly for F-theory compactifications on elliptically fibered K_3 tensored with a euclidean eight-manifold, M^8 (although more general backgrounds could also be considered). The condition that the $SL(2,\mathbb{Z})$ symmetry is unbroken is that $\delta S' = 2\pi \times k$ where k is an integer. As we shall see, this is only satisfied if M^8 has an Euler character that is a multiple of 24. This condition is reminiscent of the conditions for consistent compactifications of type IIA superstring theory to two dimensions [8], M-theory to three dimensions [9] and F-theory to four dimensions [10], where the consistent eight-manifolds were restricted to have Euler numbers that are positive multiples of 24 (although our anomaly considerations do not, by themselves, require positivity of the Euler number).

These F-theory compactifications raise further consistency questions concerning the cancellation of tadpoles that arise from wrapping D-branes around homology cycles. We will study this in some detail in section 4 for the example of F-theory compactified on $K_3 \times M^8$. In that case there are 24 seven-branes that wrap around M^8 and give tadpoles for $C^{(4)}$ and $C^{(0)}$ (the Ramond–Ramond, or $\mathcal{R}\otimes\mathcal{R}$, four-form and zero-form potentials). These tadpoles must be cancelled by adding a (integer) number of three-branes and D-instantons. The condition that this is possible is once again that χ be a multiple of 24. However, now χ is also required to be positive and, in fact, the number of D-instantons is precisely $\chi/24$.

Both the anomaly and the tadpole analysis also have analogues in terms of the heterotic string compactified on T^2 which will be discussed in section 5.

[1] This $U(1)$ anomaly was originally obtained in [7].

2. Fields and gauge fixing of type IIB supergravity

The covariant field equations of IIB supergravity are invariant under global $SL(2,\mathbb{R})$ transformations [3, 4]. There is a well-known problem in formulating a globally well-defined lagrangian for the IIB theory due to the presence of the self-dual five-form field strength. However, since this field does not play a rôle in the following, we can proceed as if there were a lagrangian (although the description of the anomaly does not actually require an explicit lagrangian [11]).

2.1. Symmetries and fields

The scalar field coset space $SL(2,\mathbb{R})/U(1)$ may be described in the usual manner by a zweibein, V_i^α $(\alpha, i = 1,2)$, with components

$$\frac{1}{\sqrt{\tau_2}}\begin{pmatrix} \tau_2\cos\phi + \tau_1\sin\phi & -\tau_2\sin\phi + \tau_1\cos\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{2.1}$$

which is a $SL(2,\mathbb{R})$ matrix. Introducing a complex basis

$$V_\pm^\alpha = (-2i)^{-1/2}\left(V_2^\alpha \pm iV_1^\alpha\right), \tag{2.2}$$

V may be written as

$$V_\pm^\alpha = \begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix} = (-2i\tau_2)^{-1/2}\begin{pmatrix} \bar\tau e^{-i\phi} & \tau e^{i\phi} \\ e^{-i\phi} & e^{i\phi} \end{pmatrix}. \tag{2.3}$$

The complex scalar τ parametrises the Poincaré upper-half plane and ϕ is an angular field $(0 \le \phi \le 2\pi)$. The global $SL(2,\mathbb{R})$ transformations act on the matrix V from the left (so that $V_\pm^\alpha$ transforms as a vector). The $U(1)$ subgroup of $SL(2,\mathbb{R})$ that acts locally consists of the real matrices

$$\begin{pmatrix} \cos\Sigma & -\sin\Sigma \\ \sin\Sigma & \cos\Sigma \end{pmatrix}, \tag{2.4}$$

which are rewritten in the complex basis as

$$\begin{pmatrix} e^{-i\Sigma} & 0 \\ 0 & e^{i\Sigma} \end{pmatrix}, \tag{2.5}$$

where $0 \le \Sigma \le 2\pi$ is an angle; this subgroup acts on V from the right (and thus on the index $\pm$ as $V_\pm^\alpha \to V_\pm^\alpha e^{\pm i\Sigma}$). Therefore, a general $SL(2,\mathbb{R})\times U(1)$ transformation acting on V gives

$$V_\pm'^\alpha = U^\alpha_{\ \beta} V_\pm^\beta e^{\pm i\Sigma(x)}, \tag{2.6}$$

where $U^\alpha_{\ \beta}$ is a constant $SL(2,\mathbb{R})$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.7}$$

and a, b, c and d are real numbers satisfying $ad - bc = 1$. Gauge invariance of the classical theory will allow one of the three scalars to be eliminated from $V^\alpha_\pm$.

The $SL(2,\mathbb{R})$-singlet combination

$$\begin{aligned} Q_\mu &= -i\epsilon_{\alpha\beta} V^\alpha_- \partial_\mu V^\beta_+ \\ &= \partial_\mu \phi - \frac{\partial_\mu \tau_1}{2\tau_2}, \end{aligned} \tag{2.8}$$

acts as a composite gauge potential since it transforms as $\delta Q_\mu = \partial_\mu \Sigma$ under the local $U(1)$ transformation in (2.6). The scalar fields also package together into a complex $SL(2,\mathbb{R})$-invariant combination, P^μ, defined by

$$\begin{aligned} P_\mu &= -\epsilon_{\alpha\beta} V^\alpha_+ \partial_\mu V^\beta_+ \\ &= i e^{2i\phi} \frac{\partial_\mu \tau}{2\tau_2}, \end{aligned} \tag{2.9}$$

which transforms under $U(1)$ as $P_\mu \to e^{2i\Sigma} P_\mu$. This means that its $U(1)$ charge is 2, while the complex conjugate field, P^*, has charge -2. The abelian field strength satisfies

$$F \equiv dQ = -iP \wedge P^* = \frac{i d\bar\tau \wedge d\tau}{4\tau_2^2}, \tag{2.10}$$

which follows from the definitions (2.8) and (2.9).

The other bosonic fields of type IIB supergravity are neutral under the local $U(1)$. The two components of the second-rank antisymmetric tensor potential, $A^\alpha_{\mu\nu}$, form a $SL(2,\mathbb{R})$ vector and transform as $A'^\alpha_{\mu\nu} = U^\alpha{}_\beta A^\beta_{\mu\nu}$. The real metric, $g_{\mu\nu}$, is neutral under both the $U(1)$ and the $SL(2,\mathbb{R})$ as is the real fourth-rank antisymmetric tensor potential, $C^{(4)}$.

Turning to the fermions, the two gravitini form a complex conjugate pair of spin-3/2 gravitino fields, ψ_μ and ψ^*_μ, with $U(1)$ charges $\pm 1/2$, while the two spin-1/2 dilatino fields form a complex conjugate pair of opposite chirality, λ and λ^*, with $U(1)$ charges $\pm 3/2$

$$\delta\psi_\mu = \frac{1}{2} i\Sigma\psi_\mu, \qquad \delta\lambda = -\frac{3}{2} i\Sigma\lambda. \tag{2.11}$$

2.2. $U(1)$ gauge fixing

In the classical supergravity theory the local $U(1)$ symmetry can be used to eliminate the field ϕ, one of the three scalar fields that parametrise the $SL(2,\mathbb{R})$ group manifold. This is achieved by a choice of 'gauge fixing condition' $\phi = \hat\phi(\tau)$. In the gauge fixed theory, the field $V^\alpha_\pm$ transforms under a global $SL(2,\mathbb{R})$ transformation as in (2.6), where $\Sigma(x)$ is chosen so as to reinstate the gauge fixing condition. More explicitly, under the action of A we find that

$$A V^\alpha_\pm = (-2i(A\tau)_2)^{-1/2} \begin{pmatrix} (A\tau)^* e^{i(\gamma - \hat\phi(\tau))} & A\tau e^{i(\hat\phi(\tau)-\gamma)} \\ e^{i(\gamma-\hat\phi(\tau))} & e^{i(\hat\phi(\tau)-\gamma)} \end{pmatrix}, \tag{2.12}$$

where $A\tau$ denotes the standard action of $SL(2,\mathbb{R})$ on the modular parameter τ

$$A\tau = \frac{a\tau + b}{c\tau + d} \qquad \text{where} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.13}$$

and

$$e^{i\gamma(A,\tau)} = \left(\frac{c\bar{\tau} + d}{c\tau + d}\right)^{1/2}. \tag{2.14}$$

The compensating gauge transformation is therefore

$$e^{i\Sigma_A} = e^{i(\gamma(A,\tau) + \hat{\phi}(A\tau) - \hat{\phi}(\tau))}, \tag{2.15}$$

and the scalars transform as in (2.13). In general, the compensating $U(1)$ transformation (2.15) is non-trivial, and since the local $U(1)$ acts on the fermions, they themselves transform non-trivially under $SL(2,\mathbb{R})$ transformations.

The above construction is covariant under the adjoint action of $SL(2,\mathbb{R})$ since if we gauge by an $U(1)$ subgroup that is obtained from (2.4) by conjugation by an element $M \in SL(2,\mathbb{R})$, $S'(\Sigma) = M^{-1}S(\Sigma)M$, then the action of $A \in SL(2,\mathbb{R})$ on τ' (where τ' is defined by $MV(\tau,\hat{\phi})M^{-1} = V(\tau',\hat{\phi}')$) is given as

$$\tau' \mapsto \frac{a'\tau' + b'}{c'\tau' + d'} \qquad \text{where} \qquad MAM^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{2.16}$$

3. The anomaly

A striking feature of type IIB supergravity (and superstring theory) is the fact that all the local gravitational anomalies cancel [11]. However, the question of anomalies in the (chiral) $U(1)$ symmetry requires separate attention. A space-time dependent $U(1)$ anomaly would be a disaster since it would prevent the elimination of the redundant bosonic field and hence would change the particle content of the theory. One way to see that this cannot happen is to use the method of descent [12, 13, 14, 15, 16] from twelve dimensions together with the fact that there is no appropriate invariant twelve-form [17]. The obvious candidate twelve-forms would have been $\mathrm{tr}R^4 \wedge F^2$ and $(\mathrm{tr}R^2)^2 \wedge F^2$, where the curvature two-form is defined in the usual manner. However, due to the fact that $F = -iP \wedge P^*$, these expressions vanish identically. The analysis of the anomaly below will show directly how the potential local anomaly is cancelled by a local counterterm.

To begin with, consider the variation of the generating function under a local $U(1)$ transformation, which may lead to a non-zero divergence of the $U(1)$ current

$$J^\mu = \frac{1}{2}\bar{\psi}_\nu \gamma^{\nu\mu\rho}\psi_\rho + \frac{3}{2}\bar{\lambda}\gamma^\mu\lambda\,. \tag{3.1}$$

At the linearized level the one-loop diagrams that can give anomalous results must have at least six vertices. The particles circulating around the diagrams are the chiral fermion fields, ψ_μ and λ. Since the chiral ten-dimensional self-dual antisymmetric tensor is neutral under the $U(1)$ it does not contribute to the anomaly. In a general gauge there may be contributions from diagrams with four external gravitons one P line and one P^* line in addition to the vertex coupling to $\partial_\mu J^\mu$; this is a *heptagon* diagram. Since there is a vertex in the theory that couples a P to a ψ and λ field such diagrams are rather complicated. There are also familiar hexagon diagrams with $\partial_\mu J^\mu$ coupling to four external gravitons and one external Q vertex, which couples to the $U(1)$ charge of the chiral fermions. Apart from the fact that Q_μ is a composite gauge field the structure of these diagrams is very similar to the mixed anomaly diagrams considered in [11]. The complete anomaly is given by the sum of diagrams with all permutations of the external legs.

Things simplify in the Feynman gauge for the gravitino (which is the gauge implicitly used in [11] and formulated more explicitly in [18]). This is the gauge in which the gravitino propagator is $\Delta_{\mu\nu} = \delta_{\mu\nu}(\gamma \cdot p)^{-1}$ and in which there is a bosonic spin-$\frac{1}{2}$ ghost field that has the opposite chirality but the same $U(1)$ charge as the gravitino. In this gauge the heptagon diagrams do not contribute to the anomaly and the only contributions come from the expected hexagon diagrams.

The calculation is standard and leads to an anomalous phase of the variation of the partition function under a local $U(1)$ transformation which is given by

$$\Delta = -\frac{1}{(2\pi)^5}\int d^{10}x \left(\frac{3}{2}\hat{I}_{1/2}\,(R, 3F/2) - \frac{1}{2}\hat{I}_{3/2}^{(D=10)}\,(R, F/2)\right)\Sigma(x)\,, \qquad (3.2)$$

where the coefficients arise from the $U(1)$ charges that enter at two vertices in the hexagon diagram and $\hat{I}$ is defined as in [11]. The integral is a linear combination of the index of the twisted Dirac operator acting on a charge-3/2 Dirac fermion (with density $\hat{I}_{1/2}(R, 3F/2)$) and on a charge-1/2 Rarita–Schwinger fermion (with density $\hat{I}_{3/2}^{(D=10)}(R, F/2)$), which is defined to include the contribution of the Fadeev–Popov ghost and where the superscript indicates the dimension of the tangent space). Since the integration is ten-dimensional and since $F^n \equiv 0$ for $n > 1$, only ten-forms linear in F in the expansion contribute

$$\begin{aligned}\hat{I}_{1/2}(R, F) &\equiv \hat{I}_{1/2}(R) \wedge F \\ &= \frac{1}{5760}(7p_1^2 - 4p_2) \wedge F\,,\end{aligned} \qquad (3.3)$$

where the p_n are the usual Pontryagin polynomials,

$$p_1 = \sum x_i^2 = -\frac{1}{2}\mathrm{tr}R^2, \qquad p_2 = \sum_{i<j} x_i^2 x_j^2 = -\frac{1}{4}\mathrm{tr}R^4 + \frac{1}{8}(\mathrm{tr}R^2)^2 \qquad (3.4)$$

and x_i are the skew eigenvalues of the curvature two form, which is a $SO(10)$ matrix. The expression $\hat{I}_{1/2}(R)$ is the eight-form contribution to the Dirac genus. The contribution from the chiral gravitino field combined with its bosonic ghost (of the same

$U(1)$ charge) is given by [11] (taking care to set $D = 10$ in the dimension-dependent term)

$$\begin{aligned}
\hat{I}_{3/2}^{(D=10)}(R,F) &\equiv \hat{I}_{3/2}^{(D=10)}(R) \wedge F \\
&= \frac{1}{5760}(303p_1^2 - 996p_2) \wedge F \,. \qquad (3.5)
\end{aligned}$$

Substituting these expressions into (3.2) and reexpressing the p_i in terms of the curvatures gives the local $U(1)$ anomaly in the form

$$\begin{aligned}
\Delta &= -\frac{1}{(2\pi)^5}\int d^{10}x \left(\frac{9}{4}\hat{I}_{1/2} - \frac{1}{4}\hat{I}_{3/2}^{(D=8)}\right) \wedge F\,\Sigma(x) \\
&= \frac{1}{(2\pi)^5}\frac{1}{96}\int (4p_2 - p_1^2) \wedge F\,\Sigma(x) \\
&= -\frac{1}{(2\pi)^5}\frac{1}{96}\int d^{10}x \left(\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2\right) \wedge F\,\Sigma(x) \,. \qquad (3.6)
\end{aligned}$$

It is quite striking that the eight-form part of the anomaly in (3.6) is proportional to the eight-form that arises as the coefficient of a $B_{\mu\nu}$ tadpole in the type IIA theory [19]. Superficially, the origin of the eight-form in (3.6) in the present calculation is entirely different from its origin in the IIA theory where it can be obtained, via the anomaly inflow argument [20], from the cancellation of a chiral gravitational anomaly in the five-brane of the type IIA theory [21]. That local anomaly arises from a sum of terms due to the presence of two six-dimensional chiral fermions and one six-dimensional self-dual antisymmetric tensor potential. The anomaly due to the two fermions descends from the eight-form $2\hat{I}_{1/2}$ while that of the self-dual antisymmetric tensor descends from the eight-form $\hat{I}_A$ that is related to the signature of the manifold. The total anomaly in the five-brane therefore descends from the expression

$$-\frac{1}{(2\pi)^4}(2\hat{I}_{1/2} + \hat{I}_A) \,. \qquad (3.7)$$

It is possible to eliminate $\hat{I}_A$ in favour of the index of the spin-3/2 operator by using the identity $21\hat{I}_{1/2} - \hat{I}_{3/2}^{(D=6)} + 8\hat{I}_A = 0$ which was noted in the discussion of gravitational anomalies in [11], for the case in which the dimension-dependent term in $\hat{I}_{3/2}$ is set to $D = 6$ (the same dimensionality as the five-brane world volume). However, in our context the index $\hat{I}_{3/2}^{(D=10)}$ in (3.5) is defined with $D = 10$ and the identity becomes $25\hat{I}_{1/2} - \hat{I}_{3/2}^{(D=10)} + 8\hat{I}_A = 0$. Using this to eliminate $\hat{I}_A$ from (3.7) gives the expression

$$\frac{1}{(2\pi)^4 \cdot 8}\left(9\hat{I}_{1/2} - \hat{I}_{3/2}^{(D=10)}\right) \equiv -\frac{X_8}{4} \,, \qquad (3.8)$$

that enters as a factor in (3.6).

The above anomaly calculation is based on the weak coupling IIB Feynman rules, but since the value of the anomaly does not depend on the coupling constant it presumably has validity beyond perturbation theory. In fact, since the value of the anomaly

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depends on F which is a function of the scalar field τ that vanishes for constant τ, non-zero values for the anomaly generically arise from configurations in which τ cannot be restricted to small coupling.

3.1. Local $U(1)$ anomaly cancellation

As expected, the local $U(1)$ anomaly can be cancelled by adding a local term to the action

$$S' = \frac{1}{(4\pi)} \int d^{10}x\, \phi\, F \wedge X_8(R)\,, \tag{3.9}$$

where X_8 is the eight-form defined by (3.8),

$$X_8 = \frac{1}{(2\pi)^4} \frac{1}{48} \left(\mathrm{tr}\, R^4 - \frac{1}{4} (\mathrm{tr}\, R^2)^2 \right) \tag{3.10}$$

and in our conventions the partition function is defined by the functional integral over the fields Φ

$$Z = \int D\Phi\, e^{iS[\Phi]}\,. \tag{3.11}$$

There is an ambiguity in the definition of (3.9) since ϕ is a periodic variable of period 2π. The consistent interpretation of the functional integral in the presence of this term therefore requires

$$\frac{1}{4\pi} \int d^{10}x\, F \wedge X_8(R) \in \mathbb{Z}\,, \tag{3.12}$$

a condition which, as we will see shortly, is automatically satisfied by consistent backgrounds.

Under a local $U(1)$ transformation

$$\phi(x) \to \phi(x) + \Sigma(x)\,, \tag{3.13}$$

while the other fields in S' are inert. Therefore $\delta S' = -\Delta$ and the local anomaly is cancelled as required. However, the term S' is not $SL(2,\mathbb{R})$ invariant since it follows from (2.12) and (2.14) that the field ϕ transforms as

$$\delta\phi = -\frac{i}{2} \ln \left(\frac{c\tau + d}{c\bar{\tau} + d} \right), \tag{3.14}$$

under $SL(2,\mathbb{R})$ while the other fields in S' are inert. The corresponding anomaly as well as the expression in (3.12) vanish automatically provided that the scalar fields are constant throughout space since then $F = 0$. The interesting configurations thus only arise for $F \neq 0$ which is characteristic of F-theory compactifications.

3.2. $SL(2,\mathbb{Z})$ anomaly cancellation

A priori one should not expect that the quantum theory maintains the whole $SL(2,\mathbb{R})$ symmetry, but only that an $SL(2,\mathbb{Z})$ subgroup is unbroken. The anomaly in this subgroup can be (partially) cancelled by the inclusion of a term of the form[2]

$$S'' = \frac{i}{4\pi}\int d^{10}x \ln g(\tau) F \wedge X_8(R)\,, \tag{3.15}$$

where $g(\tau)$ satisfies

$$g(A\tau) = U_A\left(\frac{c\tau+d}{c\bar{\tau}+d}\right)^{1/2} g(\tau)\,, \tag{3.16}$$

A is the $SL(2,\mathbb{Z})$ transformation that is defined as in (2.13), and U_A is a constant phase. Under an $SL(2,\mathbb{Z})$ transformation the term S'' therefore transforms as

$$\delta_A S'' = -\delta_A S' + i\ln U_A \frac{1}{4\pi}\int d^{10}x\, F\wedge X_8(R)\,. \tag{3.17}$$

The anomaly would be cancelled completely if it were possible to have $U_A = 1$ for all $A \in SL(2,\mathbb{Z})$, but as we shall demonstrate momentarily, this is not possible. As a consequence only the τ-dependent term in the anomaly is removed by this term, and the condition that $SL(2,\mathbb{Z})$ is unbroken leads to a non-trivial constraint on allowed compactifications.

The group of $SL(2,\mathbb{Z})$ is generated by the two transformations $T:\tau\mapsto\tau+1$ and $S:\tau\mapsto -1/\tau$ subject to the relations $S^2 = 1 = (TS)^3$. In terms of these generators, the condition (3.16) becomes

$$\begin{aligned} g(\tau+1) &= U_T\, g(\tau) \\ g(-1/\tau) &= U_S\,\left(\frac{\tau}{\bar{\tau}}\right)^{1/2} g(\tau) \end{aligned} \tag{3.18}$$

where U_S and U_T are constant phases. Because of the square root in (3.18) there is a sign ambiguity, and we therefore only have the relations $U_S^4 = 1 = (U_T U_S)^6$. One solution to these conditions is given by

$$g_0(\tau) = \frac{\eta(\tau)}{\eta(\tau)^*}\,, \tag{3.19}$$

where η is the Dedekind eta-function

$$\eta(\tau) = e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})\,. \tag{3.20}$$

In this case

$$U_S^0 = e^{i\pi/2} \qquad U_T^0 = e^{i\pi/6} \tag{3.21}$$

since η transforms under the generators of $SL(2,\mathbb{Z})$ as

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau) \qquad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)\,. \tag{3.22}$$

[2]In a previous version of this paper, a different approach was employed which only worked in the specific situation considered there.

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The most general solution of (3.18) can then be written as $g(\tau) = g_0(\tau)\, h(\tau)$, where h satisfies

$$
\begin{aligned}
h(\tau+1) &= \hat{U}_T\, h(\tau) \\
h(-1/\tau) &= \hat{U}_S\, h(\tau)\,.
\end{aligned}
\tag{3.23}
$$

Because of the relations of $SL(2,\mathbb{Z})$, $\hat{U}_S^2 = 1$ and $(\hat{U}_T \hat{U}_S)^3 = 1$ (where there is now no sign ambiguity), and the possible values of U_S and U_T are

$$
U_S = \hat{U}_S e^{i\pi/2} = \pm e^{i\pi/2} \qquad U_T = \hat{U}_T e^{i\pi/6}\,. \tag{3.24}
$$

In particular, U_S is always a (primitive) fourth root of unity, and U_T is either a twelfth root of unity or a fourth root of unity. The former case is realised by the example (3.19) above, and an example of the latter case is given by

$$
g_1(\tau) = g_0(\tau) \left(\frac{J(\tau)^*}{J(\tau)} \right)^{\frac{1}{12}}, \tag{3.25}
$$

where J is the standard modular invariant J-function, and we have

$$
\begin{aligned}
\left(\frac{J^*(\tau+1)}{J(\tau+1)} \right)^{1/12} &= e^{2\pi i/6} \left(\frac{J^*(\tau)}{J(\tau)} \right)^{1/12} \\
\left(\frac{J^*(-1/\tau)}{J(-1/\tau)} \right)^{1/12} &= - \left(\frac{J^*(\tau)}{J(\tau)} \right)^{1/12}.
\end{aligned}
\tag{3.26}
$$

In this case U_S and U_T generate a $\mathbb{Z}_4$ subgroup of $U(1)$.

It is now clear that the $SL(2,\mathbb{Z})$ anomaly can only be cancelled up to a term of the form

$$
\frac{2\pi}{N} \frac{1}{4\pi} \int d^{10}x\, F \wedge X_8\,, \tag{3.27}
$$

where $N = 4$ if g has the same transformation properties as g_1, and $N = 12$ otherwise. In order for the $SL(2,\mathbb{Z})$ symmetry to be unbroken, (3.27) must be a multiple of 2π, and hence the compactification can only preserve the $SL(2,\mathbb{Z})$ symmetry if

$$
\frac{1}{4\pi} \int d^{10}x\, F \wedge X_8 \in 4\mathbb{Z}\,. \tag{3.28}
$$

This condition guarantees, in particular, that (3.12) is satisfied, and therefore that the counterterm S' is well-defined.

These considerations do not uniquely determine the term S''. In particular, a given g can always be multiplied by a modular invariant function $\hat{h}$ without modifying the anomaly cancellation. In principle it should be possible to determine the actual form of the term S'' from a direct string calculation, but we shall not do so here. However, if one were to assume that g should be a product of a holomorphic and anti holomorphic function and that $\ln g$ has a perturbative term that is precisely $i\pi C^{(0)}/2$ then g is uniquely determined to be g_1 defined in (3.25).

The anomaly can also be understood in the gauge fixed formulation: if we fix the gauge by setting $\phi = 0$, the counterterm S' vanishes but the fermion determinant transforms non-trivially under $SL(2,\mathbb{Z})$ transformations as follows from (2.15). This is again compensated by the anomalous transformation of S'' up to the τ-independent terms discussed above.

3.3. F-theory on K3

In order to exhibit the constraint (3.28) more explicitly, let us consider the simplest example in which it leads to a non-trivial condition. This is the theory where type IIB is compactified on the euclidean product space $S^2 \times M^8$ that arises by inserting 24 seven-branes with locations at points in S^2. The world-volumes of the branes are wrapped around the euclidean eight-manifold M^8 (we are here envisioning a Wick rotation of the signature of both the target space and the eight-volumes of the seven-branes). In terms of F-theory this is the compactification on $K3 \times M^8$. In order for this compactification to preserve supersymmetry, M^8 has to be a spin-manifold with a nowhere vanishing chiral spinor.

In this case, using the fact that $\frac{1}{4\pi}\int F = 1$ [22], the consistency condition (3.28) simply becomes

$$\frac{1}{4}\int d^8x\, X_8(R) = \frac{\chi}{24} \in \mathbb{Z}\,, \tag{3.29}$$

where χ is the Euler characteristic of the M^8 and we have used the relation

$$\hat{\chi} = \frac{1}{8}(4p_2 - p_1^2)\,, \tag{3.30}$$

which is true for any eight-manifold with a nowhere vanishing spinor [23] (and $\hat{\chi}$ is the density which integrates to the Euler character, $\chi = (2\pi)^{-4}\int\hat{\chi}$). The identity (3.29) also enters in discussions of the compactification of type IIA superstrings [8], M theory [9] and F theory [10] on Calabi–Yau four-folds (CY4).

Th consistency condition (3.12) under which the $U(1)$ counterterm S' is well-defined is in this case

$$\int d^8x X_8(R) = \frac{\chi}{6} \in \mathbb{Z}\,. \tag{3.31}$$

It was shown in [10] that for the case of a Calabi-Yau 4-fold, $\chi/6$ is always an integer and therefore the consistency condition (3.12) is satisfied. It is also true that $\chi/6$ is an integer for all of the currently known $Spin(7)$ manifolds listed in [24, 25]. It would be of interest if this could be proven to be true for all manifolds with a nowhere vanishing spinor.

4. Tadpole analysis

It is instructive to contrast the above calculation with a tadpole analysis of the F-theory compactification. The simplest examples of such compactifications [26] can be

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described by the insertion of 24 parallel seven-branes in the type IIB vacuum. Of these, up to 18 may be $(1,0)$ D7-branes while the rest must be other (p,q) seven-branes.[3]

Each of the 24 seven-branes in the F-theory compactification has a term in its world-volume action of the form [27, 28, 29]

$$-\frac{1}{48}\frac{1}{(2\pi)^2}\int C^{(4)} p_1\,, \tag{4.1}$$

where $C^{(4)}$ is the (self-dual) RR 4-form of type IIB, and p_1 is the first Pontryagin class. For every 4-cycle X_4 in M^8 for which $\int_{X_4} p_1 \neq 0$, type IIB theory has a $C^{(4)}$ tadpole on the compact submanifold X_4, and in order to cancel it, we have to include

$$N(X_4) = 24\frac{1}{48}\frac{1}{(2\pi)^2}\int_{X_4} p_1 \tag{4.2}$$

D3-branes in the vacuum that are points on X_4 and extend along the transverse directions. Since X_8 is a spin manifold, its second Stiefel-Whitney class vanishes, and therefore $p_1/2$ is an integer cohomology class of X_8;[4] this implies that $N(X_4)$ is an integer for every X_4.

We now wish to determine the $C^{(0)}$ tadpole of the resulting configuration. Without loss of generality we may take sixteen of the D7-branes to be $(1,0)$ seven-branes. Each of these has a term [28]

$$\frac{1}{(2\pi)^4}\frac{1}{(24)^2\,40}\int C^{(0)}\left(9p_1^2 - 8p_2\right), \tag{4.3}$$

in its worldvolume action. Since $C^{(0)}$ is not invariant under $SL(2,\mathbb{Z})$, it is not immediately clear what the contribution of the remaining eight seven-branes is. However, following Sen [8] we can think of these eight seven-branes as corresponding to four orientifold seven-planes that have each split into two seven-branes. At least at the point in moduli space where these two seven-branes are on top of each other, we can describe them in terms of orientifold planes, and we can thus summarize their gravitational effects by a world-volume action of the form [31]

$$\frac{1}{(2\pi)^4}\frac{1}{(24)^2\,20}\int C^{(0)}\left(27p_1^2 - 44p_2\right). \tag{4.4}$$

Taking the contributions of the sixteen D7-branes (4.3) and the four orientifold seven-planes (4.4) together, the total contribution of the 24 seven-branes to the $C^{(0)}$ tadpole becomes

$$\begin{aligned} I_7 &= \frac{1}{(2\pi)^4}\int_{M^8} C^{(0)}\Big(\frac{1}{192}(p_1^2 - 4p_2) + \frac{1}{96}p_1^2\Big) \\ &= -\frac{\chi}{24}C^{(0)} + \frac{1}{(2\pi)^4}\int_{M^8}\frac{p_1^2}{96}C^{(0)}, \end{aligned} \tag{4.5}$$

[3] The coprime integers p and q label the $NS \otimes NS$ and $\mathcal{R} \otimes \mathcal{R}$ charges of the D-strings that can end on the various seven-branes.

[4] The reduction of the first Pontryagin class p_1 mod 2 is precisely the square of the second Stiefel-Whitney w_2 class on X_8, and thus vanishes for spin manifolds [30].

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where we have again used the identity $\hat{\chi} = 8(4p_2 - p_1^2)$. We also have to include the $C^{(0)}$ tadpole contribution of the D3-branes, each of which has the term

$$-\frac{1}{(2\pi)^2}\frac{1}{48}\int C^{(0)}p_1\,, \tag{4.6}$$

in their world-volume action. For every 4-cycle X_4, there are $N(X_4)$ D3-branes that extend along the transverse directions, and the total contribution of the D3-branes to the $C^{(0)}$ tadpole is therefore

$$\begin{aligned} I_3 &= -\sum_{X_4} N(X_4)\frac{1}{(2\pi)^2}\frac{1}{48}\int_{X_4^\perp} p_1 C^{(0)} \\ &= -24\frac{1}{(2\pi)^4}\frac{1}{(48)^2}\int_{M^8} p_1^2\, C^{(0)} = -\frac{1}{96}\int_{M^8} p_1^2\, C^{(0)}\,. \end{aligned} \tag{4.7}$$

Thus taking I_7 and I_3 together, the total $C^{(0)}$ tadpole is $-\chi/24\ C^{(0)}$, which can be cancelled by the inclusion of $\chi/24$ D-instantons, provided that χ is again divisible by 24.

As explained in [10], the configuration breaks supersymmetry unless the D-branes that are added to the background in order to cancel the tadpoles are branes rather than anti-branes. This leads to a further restriction on the signs of (4.2) and χ. It is difficult to establish these signs from first principles, but we can determine them by considering special examples. For example, the compactification on $K3 \times K3$ preserves supersymmetry, and since $\frac{1}{(2\pi)^2}\int_{K3} p_1 = -48$ and $\chi_{K3\times K3} = 24^2$, we deduce that χ has to be a positive integer and $N(X_4)$ a negative integer.

The above analysis applies to an arbitrary 8-dimensional manifold that has a covariantly constant spinor, including Calabi–Yau four-folds and $Spin(7)$ manifolds. The $Spin(7)$ manifolds constructed by Joyce [24] all have Euler character $\chi = 6 \cdot 24$ (and signature $\sigma = 64$), and thus satisfy the above condition. Additional $Spin(7)$ manifolds have been found for which the signature is different from 64, but the Euler character of all known examples is divisible by 24 [25]. These examples all arise as blow-ups of (symmetric) geometrical toroidal orbifolds, and one may expect that the corresponding string theory (or F-theory) is necessarily consistent. This would 'explain' why the Euler character of all of these examples is divisible by 24. This explanation should also apply in the case of compactifications on Calabi–Yau four-folds, which generically have Euler characters that are multiples of 6 only [10]. There again, the consistency of string compactifications should imply that any four-fold that is constructed by blowing up a symmetric orbifold should have an Euler number that is a multiple of 24.

5. Relation to the heterotic string

The compactification of F theory on (elliptically fibred) K_3 is supposed to be equivalent to the heterotic string compactified on T^2 [26], and thus the same F-theory features

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that we have discussed should be apparent from the point of view of the heterotic string. The eight-dimensional theory has a duality symmetry group which contains $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$, where the two $SL(2,\mathbb{Z})$ are associated with transformations of the Kähler class and the complex structure of the two-torus, T^2. For comparison with the F-theory discussion it is sufficient to consider generic Wilson lines on T^2 which break the heterotic gauge symmetry to $U(1)^{20}$. The eight-dimensional low energy action, which is the Einstein–Maxwell action of [32, 33] with eighteen vector multiplets, has chiral couplings that give rise to an anomaly in a $U(1)$ current that is embedded in an $SL(2,\mathbb{R})$ subgroup of the classical symmetry group of the theory. This $U(1)$ anomaly of the eight-dimensional supergravity theory is essentially the same anomaly that we discussed earlier in a somewhat different F-theory guise. As before, the anomaly can be expressed as a local $U(1)$ anomaly together with an anomaly cancelling term that does not preserve $SL(2,\mathbb{Z})$. In this case the local anomaly has the form

$$\int d^8x \left(X_8 + 12 I_{1/2}\right) \Sigma(x) \tag{5.1}$$

and the anomaly cancelling term has a form analogous to (3.9). This term again induces an $SL(2,\mathbb{Z})$ anomaly that can be partially cancelled by a term analogous to (3.15). When the theory is compactified on an eight-manifold M^8, the whole of $SL(2,\mathbb{Z})$ is unbroken provided that $\int X_8/4 = \chi/24$ is again an integer; since $\int 3I_{1/2}$ is always integer, the second term in (5.1) does not affect the result.

Another problem with the eight-dimensional heterotic string, arises from the well-known term that is associated with the cancellation of ten-dimensional gravitational (and gauge) anomalies. After compactification on T^2 it has the form $\int d^8x\, BY_8(R)$, where $B \equiv B_{9\,10}$ and [34]

$$Y_8 = \frac{1}{4}X_8 + 15 I_{1/2} = \frac{1}{128(2\pi)^4}\left(\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2\right). \tag{5.2}$$

Upon further compactification on M^8 this gives rise to a B tadpole which is the same as the $C^{(0)}$ tadpole arising from the compactification of the 24 F-theory seven-branes in the last section. This tadpole is cancelled by wrapping the world-volumes of $N(X_4)$ five-branes around the torus and the transverse four-cycles in M^8, and by wrapping the world-sheets of $\chi/24$ fundamental strings around T^2. Again this is only possible if $\chi/24$ is a positive integer.

To summarize, we have seen that type IIB supergravity has a local $U(1)$ anomaly that can be cancelled by a local counterterm which induces an anomaly in the global $SL(2,\mathbb{R})$ transformations; this means that $SL(2,\mathbb{R})$ is not a symmetry of the theory. However, the τ-dependence of the anomaly in an $SL(2,\mathbb{Z})$ subgroup is cancelled by the inclusion of another term (S'') and the remaining τ-independent anomaly is irrelevant in the path-integral provided that the background satisfies a certain topological restriction. For F-theory compactifications of the form $K3 \times M^8$, the requirement is that the Euler character of the eight-dimensional manifold is a multiple of 24. This

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condition is compatible with (although slightly weaker than) the condition that arises from the requirement of tadpole cancellation in F-theory compactifications. Similar considerations also apply to the dual heterotic string theory.

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